Last time: given a set S (alphabet) and a set R of words in the letters of S

relations

one can define the group $\langle S \mid R \rangle$ in two ways

group with generators S and relations R

- Explicit: $\langle S|R\rangle = F_S / (\text{smallest normal subgroup containing } R)$

free group of words $s_1^{\pm 1} \dots s_k^{\pm 1}$, $s_1, \dots, s_k \in S$ with operation given by concatenation

- Implicit: $\langle S|R \rangle$ is the unique group (up to isomorphism) such that the following universal property holds

$\exists$ 1-to-1 correspondence $\forall$ group G

$$\{\text{functions } S \xrightarrow{\alpha} G \mid \alpha(R) = e\} \xleftrightarrow{\Psi_{S|R,G}} \{\text{homs } \langle S|R\rangle \xrightarrow{\beta} G\}$$

such that the following diagram commutes

$$\{\text{functions } S \xrightarrow{\alpha} G \mid \alpha(R) = e\} \xleftrightarrow{\Psi_{S|R,G}} \{\text{homs } \langle S|R\rangle \xrightarrow{\beta} G\}$$

β

$\alpha' = g \circ \alpha \circ f$ $\qquad\qquad\qquad\qquad$ $\square$ $\qquad\qquad\qquad\qquad$ $\beta' = g \circ \beta \circ f$

$$\left\{ \text{functions } S' \xrightarrow{\alpha'} G' \;\middle|\; \alpha'(R') = e' \right\} \xleftrightarrow{\Psi_{S'|R',G'}} \left\{ \text{homs } \langle S' | R' \rangle \xrightarrow{\beta'} G' \right\}$$

$\forall$ homomorphism $G \xrightarrow{g} G'$ and $\forall$ function $S' \xrightarrow{f} S$ such that $f(r')$ = concatenation of words in R and their inverses, $\forall\, r' \in R'$

Application: free products of groups (with amalgamation)
(important in Seifert-van Kampen theorem in topology)

Def: suppose you have groups and homomorphisms

$$\begin{tikzcd} \langle S | R \rangle = G & & G' = \langle S' | R' \rangle \\ & H \arrow[ul, "f"] \arrow[ur, "f'"'] & \end{tikzcd}$$

Their free product with amalgamation is

assume disjoint, even if $G = G'$

$$G *_H G' = \langle S \sqcup S' \mid R \sqcup R' \sqcup f(h) f'(h)^{-1} \rangle_{\forall h \in H}$$

(if $H = 1$, this is just called free product)

$H \xrightarrow{f} G$

g'

$g(s_1^{\pm 1} \dots s_k^{\pm 1}) = s_1^{\pm 1} \dots s_k^{\pm 1}$

$g'(s_1'^{\pm 1} \dots s_k'^{\pm 1}) = s_1'^{\pm 1} \dots s_k'^{\pm 1}$

$$\begin{tikzcd} \arrow[d, "f"'] \arrow[r, phantom, "\square"] & \arrow[d, "g'"] \\ G' \arrow[r, "g"'] & G \underset{H}{*} G' \end{tikzcd}$$

Note: the fact that $f(h)f'(h)^{-1} = e$ in $G \underset{H}{*} G'$ $\Leftrightarrow$ $f \circ g = f' \circ g'$

Prop: up to isomorphism, $G \underset{H}{*} G'$ does not depend on the choice of S, R, S', R' in $G \cong \langle S | R \rangle$, $G' \cong \langle S' | R' \rangle$

Proof:

$$\left\{ s : G \underset{H}{*} G' \xrightarrow{\text{hom}} A \right\}$$

$$\updownarrow$$

$$\left\{ \alpha : S \sqcup S' \xrightarrow{\text{fcts}} A \;\middle|\; \alpha(r) = e,\ \alpha(r') = e,\ \alpha(f(h)f'(h)^{-1}) = e,\ \forall h \in H \right\}$$

$$\updownarrow$$

$$\left\{ \begin{array}{l} \alpha : S \xrightarrow{\text{fct}} A \\ \alpha' : S' \longrightarrow A \end{array} \;\middle|\; \begin{array}{l} \alpha(r) = e \\ \alpha(r') = e \end{array},\ \alpha(f(h)) = \alpha'(f'(h)),\ \forall h \in H \right\}$$

$$\updownarrow$$

$$\left\{ \begin{array}{l} \beta : \underset{\substack{\shortparallel \\ G}}{\langle S | R \rangle}\!\!\!\!\!\!\!\!\!\!\!\!\overset{G}{} \longrightarrow A \\ \beta' : \underset{\substack{\shortparallel \\ G'}}{\langle S' | R' \rangle} \longrightarrow A \end{array} \;\middle|\; \beta(f(h)) = \beta'(f'(h)),\ \forall h \in H \right\}$$

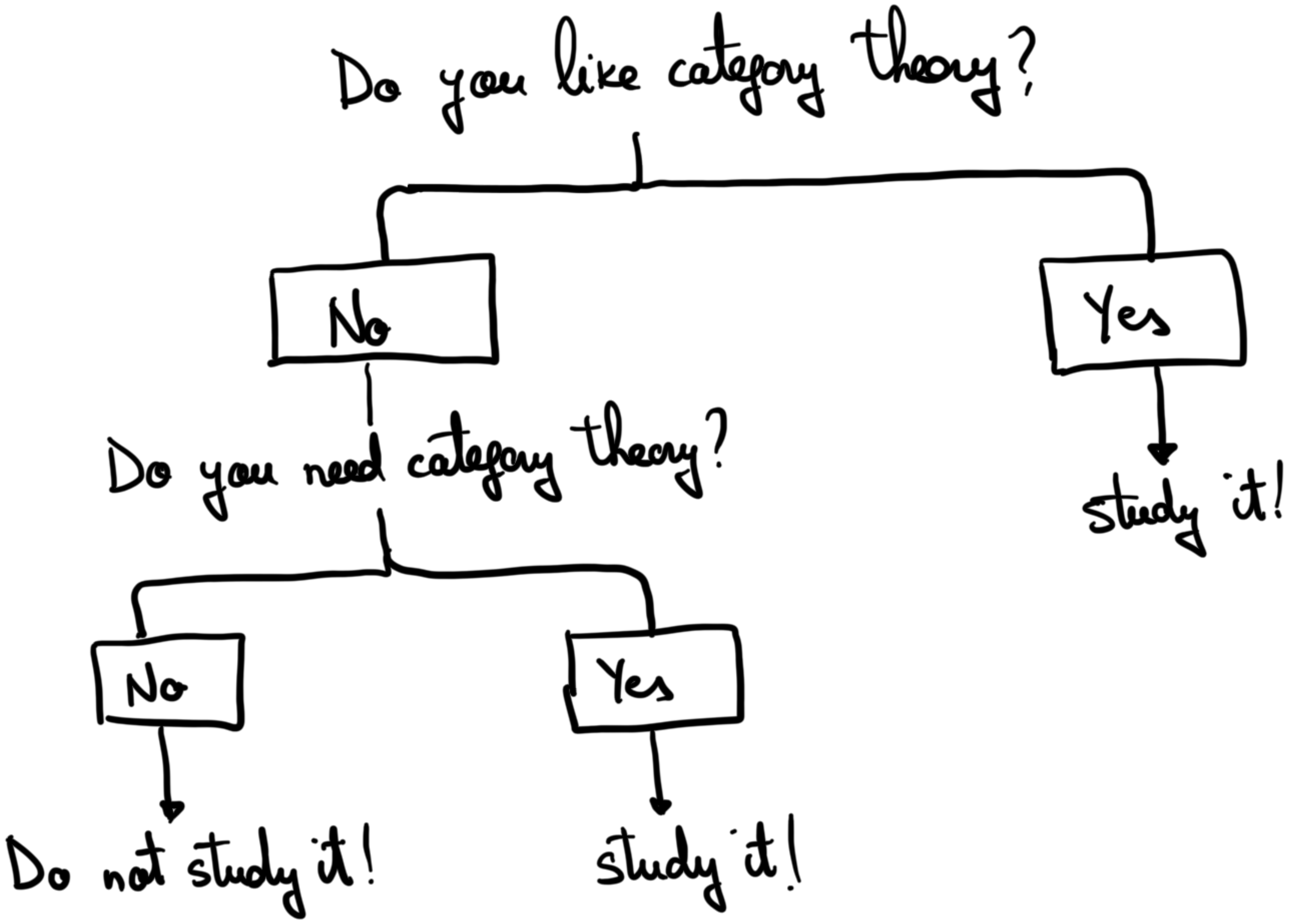
$$\updownarrow$$

$$\left\{ \beta, \beta' \text{ in } \begin{tikzcd} H \arrow[r, "f'"] \arrow[d, "f"'] & G' \arrow[d, "\beta'"] \\ {} & {} \end{tikzcd} \;\middle|\; \beta \circ f = \beta' \circ f' \right\}$$ no more mention of S, R, S', R'

$G \xrightarrow{\beta} A$

Category theory is not a theory; it's a language.

Should I study category theory?



Def: a category $\mathcal{C}$ consists of the following data

- a set* of objects $Ob(\mathcal{C})$
- $\forall X, Y \in Ob(\mathcal{C})$ a set* of morphisms $Mor_{\mathcal{C}}(X,Y)$

• $\forall X, Y \in Ob(\mathcal{C})$, a set of

together with a notion of composition

$$\boxed{Mor_{\mathcal{C}}(X,Y) \times Mor_{\mathcal{C}}(Y,Z) \longrightarrow Mor_{\mathcal{C}}(X,Z)}$$

$$\underset{f}{\rotatebox{90}{$\in$}} \qquad\qquad \underset{g}{\rotatebox{90}{$\in$}} \quad \rightsquigarrow \quad \underset{g\circ f}{\rotatebox{90}{$\in$}}$$

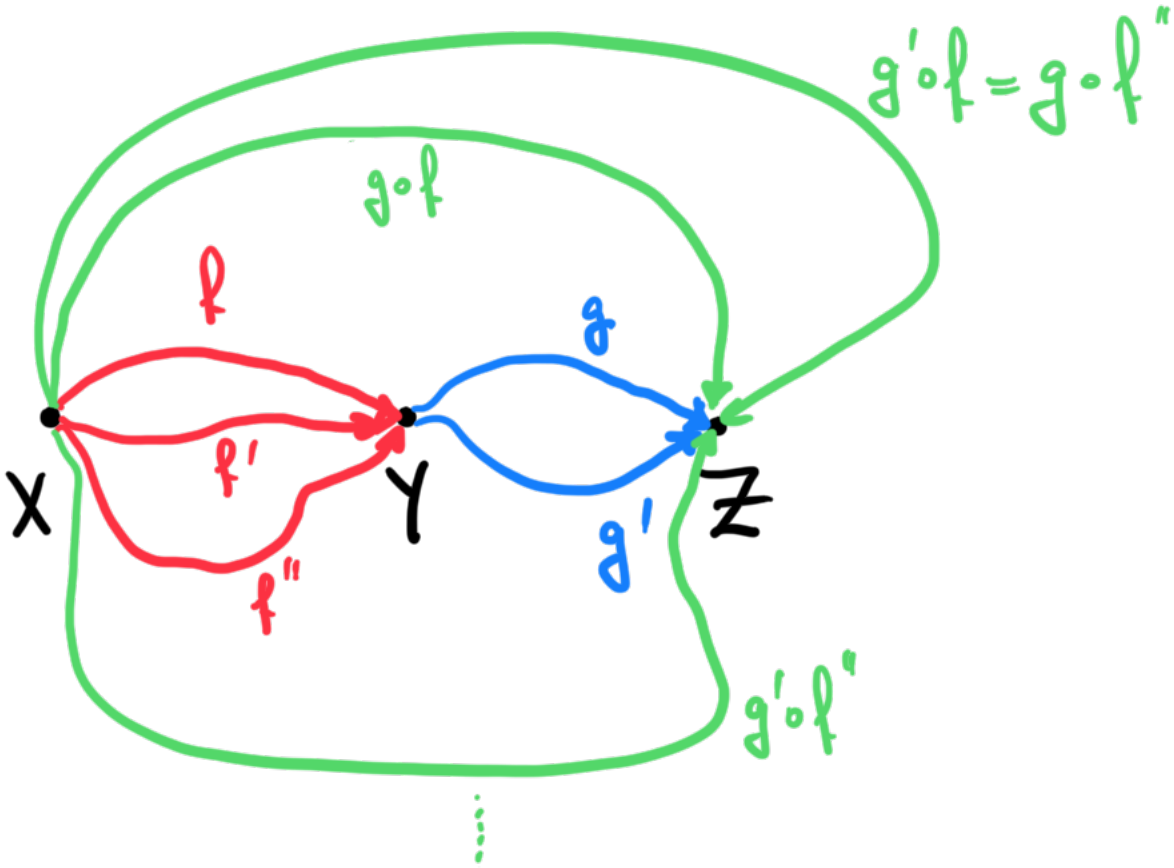
which satisfies a bunch of axioms:

(1) $\exists$ specials $Id_X \in Mor_{\mathcal{C}}(X,X)$ $\forall X \in Ob(\mathcal{C})$ s.t.

$$Id_Y \circ f = f \circ Id_X = f \quad , \forall f \in Mor_{\mathcal{C}}(X,Y)$$

(2) $h \circ (g \circ f) = (h \circ g) \circ f$, $\quad \forall f \in Mor_{\mathcal{C}}(X,Y)$
$\forall g \in Mor_{\mathcal{C}}(Y,Z)$
$\forall h \in Mor_{\mathcal{C}}(Z,T)$

Visualization : oriented graph where vertices are objects and arrows are morphisms



(1)

$$\begin{tikzcd} X \arrow[loop above, "Id_X"] \arrow[r, "f"'] & Y \arrow[loop above, "Id_Y"] \end{tikzcd}$$

$$Id_Y \circ f = f \circ Id_X = f$$

(2)

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[rr, bend left, "g\circ f"] \arrow[rrr, bend left=60, "h\circ(g\circ f)"] \arrow[rrr, bend right=70, "(h\circ g)\circ f"'] & Y \arrow[r, "g"'] \arrow[rr, bend right, "h\circ g"'] & Z \arrow[r, "h"'] & T \end{tikzcd}$$

blue

||

green

Write $f : X \longrightarrow Y$ instead of $f \in Mor_{\mathcal{C}}(X, Y)$

Ex: $\mathcal{C} = \mathbf{Set}$ whose objects are sets and whose morphisms are functions between sets

$\mathcal{C} = \mathbf{Gr}$ whose objects are groups and whose morphisms are homomorphism between groups

Def: the inverse of a morphism $f: X \to Y$ in a category $\mathcal{C}$ is a morphism $g: Y \to X$ s.t.

if it exists, it is unique

$$\boxed{f \circ g = Id_Y \text{ and } g \circ f = Id_X}$$

(a morphism which has an inverse is called a isomorphism)

Ex: a category $\mathcal{C}$ with a single object • and all morphisms invertible

$\Updownarrow$

a group $Mor_{\mathcal{C}}(\cdot, \cdot)$

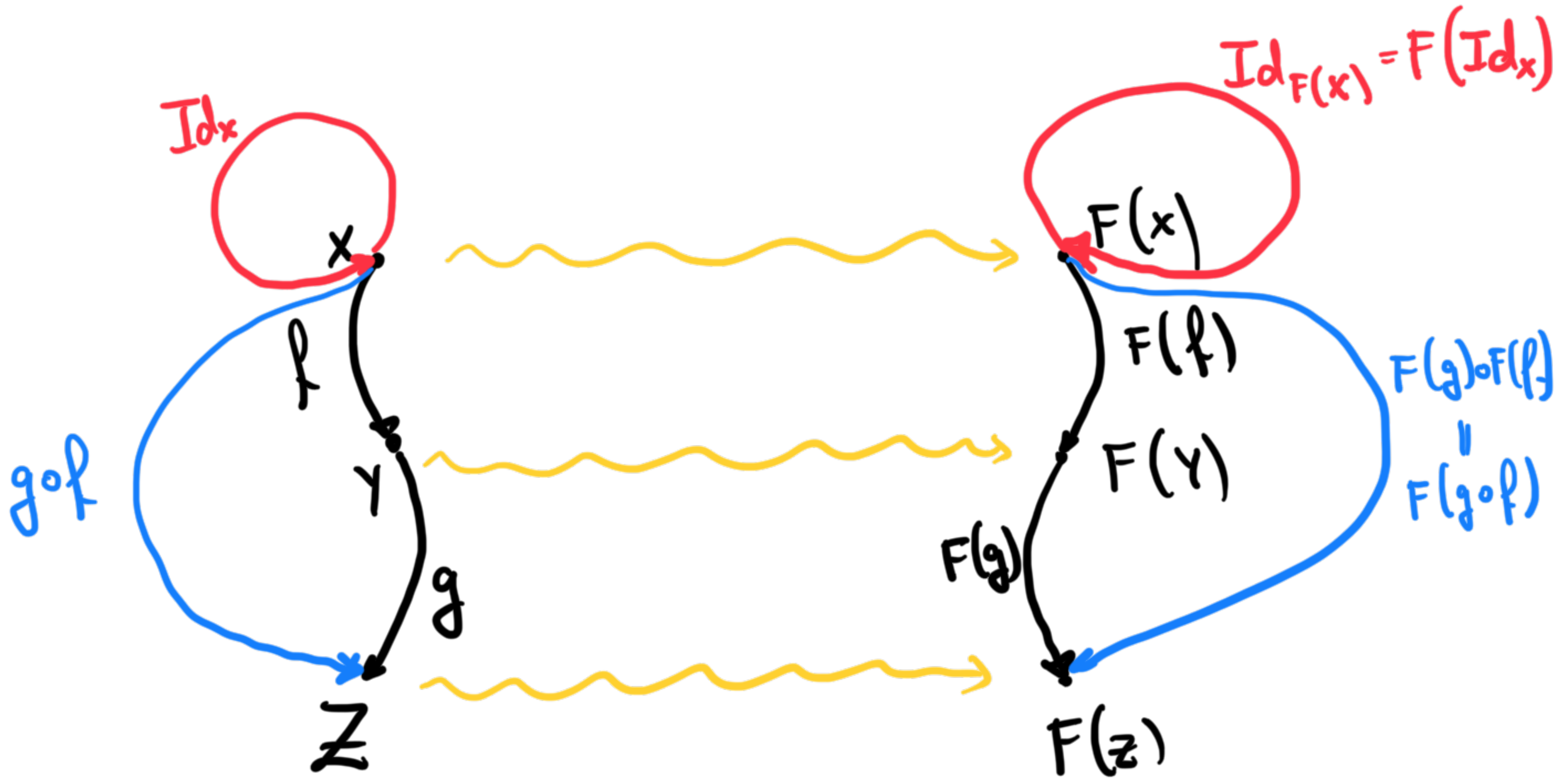
Def, a functor $F: \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ consists of data

- a function $F: Ob(\mathcal{C}) \to Ob(\mathcal{D})$
  $X \in Ob(\mathcal{C}) \rightsquigarrow F(X) \in Ob(\mathcal{D})$
- an assignment $f: X \to Y \rightsquigarrow F(f): F(X) \to F(Y)$

in $\mathcal{C}$ in $\mathcal{D}$

satisfying a bunch of compatibility conditions with respect to identity morphisms and compositions (below)

Visualisation: $\mathcal{C} \xrightarrow{F} \mathcal{D}$



Ex: The forgetful functor

$$Gr \xrightarrow{for} Set$$

$$for(G) = (\text{underlying set of } G)$$

$$for\begin{pmatrix} G \\ \downarrow \text{hom} \\ G' \end{pmatrix} = \begin{pmatrix} \text{underlying set of } G \\ \downarrow \text{function} \\ \text{underlying set of } G' \end{pmatrix}$$

The free group functor

$$\mathrm{Set} \xrightarrow{\text{free}} \mathrm{Gr}$$

$$\text{free}(S) = F_S$$

$$\text{free}\begin{pmatrix} S \\ \downarrow f \\ T \end{pmatrix} = \begin{pmatrix} F_S \\ \downarrow \phi \\ F_T \end{pmatrix} \quad \text{where } \phi(s_1^{\pm 1} \dots s_k^{\pm 1}) = f(s_1)^{\pm 1} \dots f(s_k)^{\pm 1}$$

Def: functors $G : \mathcal{C} \longrightarrow \mathcal{D}$

$\mathcal{C} \longleftarrow \mathcal{D} : F$

are called **adjoint** if $\exists$ 1-to-1 correspondences $\forall X \in Ob(\mathcal{D})$, $\forall Y \in Ob(\mathcal{C})$

$$\mathrm{Mor}_{\mathcal{C}}(F(X), Y) \overset{\Psi_{X,Y}}{\longleftrightarrow} \mathrm{Mor}_{\mathcal{D}}(X, G(Y))$$

which are natural (functorial), i.e. the following commutes

$$\begin{tikzcd}
\begin{smallmatrix} F(X) \\ \downarrow \alpha \\ Y \end{smallmatrix} \in \mathrm{Mor}_{\mathcal{C}}(F(X), Y) \arrow[r, leftrightarrow, "\Psi_{X,Y}"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Mor}_{\mathcal{D}}(X, G(Y)) \ni \begin{smallmatrix} X \\ \downarrow \alpha \\ G(Y) \end{smallmatrix} \arrow[d] \\
\begin{smallmatrix} F(X') \\ \downarrow F(f) \\ F(X) \end{smallmatrix} \quad \mathrm{M}\ldots(F(X'), Y') \arrow[r, leftrightarrow, "\Psi_{X',Y'}"] & \mathrm{M}\ldots(X', G(Y')) \ni \begin{smallmatrix} X' \\ \downarrow f \\ X \\ \downarrow \alpha \end{smallmatrix}
\end{tikzcd}$$

$$\begin{array}{ccc} F(X) & \in \mathrm{Mor}_{\mathcal{C}}(F(X), Y) \longleftrightarrow \mathrm{Mor}_{\mathcal{D}}(X, G(Y)) & \\ \downarrow \alpha & & G(Y) \\ Y & & \downarrow G(g) \\ \downarrow g & & G(Y') \\ Y' & & \end{array}$$

$\forall$ morphism $Y \xrightarrow{g} Y'$ in $\mathcal{C}$ and $\forall$ morphism $X' \xrightarrow{f} X$ in $\mathcal{D}$

(F is called the left adjoint of G)

(G is called the right adjoint of F)

Thm (last time) G = for : $Gr \longrightarrow Set$ (right)

$Gr \longleftarrow Set$: free $= F$ (left)

are adjoints of each other.

<u>Def</u>: in a category $\mathcal{C}$ suppose you have objects and morphisms

$$\begin{array}{ccc} X & \xrightarrow{f'} & Y' \\ f \downarrow & & \\ Y & & \end{array}$$

Their **pushout** is an object Z and morphisms g, g' as in

$$\begin{array}{ccc} X & \xrightarrow{f'} & Y' \\ f \downarrow & & \downarrow g' \end{array} \qquad \text{s.t. } g \circ f = g' \circ f'$$

$$
\begin{tikzcd}
\arrow[d, "f"'] & \arrow[d, "g'"] \\
Y \arrow[r, "g"'] & Z
\end{tikzcd}
$$

which satisfy the following universal property:

any time we have an object A and morphisms h, h' as in

$$
\begin{tikzcd}
X \arrow[r, "f'"] \arrow[d, "f"'] & Y' \arrow[ddr, bend left, "h'"] & \\
Y \arrow[drr, bend right, "h"'] & & \\
& & A
\end{tikzcd}
\qquad \text{s.t. } h \circ f = h' \circ f'
$$

then there is a unique morphism $Z \xrightarrow{s} A$ which makes all squares and triangles below commute

$$
\begin{tikzcd}
X \arrow[r, "f'"] \arrow[d, "f"'] & Y' \arrow[d, "g'"] \arrow[ddr, bend left, "h'"] & \\
Y \arrow[r, "g"'] \arrow[drr, bend right, "h"'] & Z \arrow[dr, "s"] & \\
& & A
\end{tikzcd}
\qquad \text{i.e.} \quad
\begin{aligned}
h &= s \circ g \\
h' &= s \circ g'
\end{aligned}
$$

<u>Thm</u>: if a pushout exists, then it is unique up to iso.

suppose $\exists$ two pushouts $\quad Y' \xrightarrow{g'} \quad\quad Y' \xrightarrow{\tilde{g}'}$

$Y \xrightarrow{g} Z \qquad Y \xrightarrow{\tilde{g}} \tilde{Z}$

Take $A = \tilde{Z}$ in the definition of Z being a pushout

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"'] & Y' \arrow[d, "g'"] \arrow[ddr, bend left, "\tilde{g}'"] & \\
Y \arrow[r, "g"] \arrow[drr, bend right, "\tilde{g}"'] & Z \arrow[dr, shift right, "s"'] & \\
 & & \tilde{Z} \arrow[ul, shift right, "s'"']
\end{tikzcd}$$

$\tilde{g}' = s \circ g'$

$\tilde{g} = s \circ g$

$g' = s' \circ \tilde{g}'$

$g = s' \circ \tilde{g}$

Take $A = Z$ in the definition of $\tilde{Z}$ being a push out

Finally, the uniqueness part of the definition of pushouts implies that $s \circ s' = s' \circ s = Id \implies Z \cong \tilde{Z}$ $\square$

Ex: in the category Gr,

pushouts = free products with amalgamation

$$\begin{tikzcd}
H \arrow[r, "f'"] \arrow[d, "f"'] & G' \arrow[d, "g'"] \arrow[dr, bend left, "h'"] & \\
 & &
\end{tikzcd}$$

$$
\begin{tikzcd}
G \arrow[r, "g"'] \arrow[rrd, bend right, "h"'] & G \underset{H}{*} G \arrow[rd, "s"] & \\
& & A
\end{tikzcd}
$$

Proof: $\left\{ s : G \underset{H}{*} G' \xrightarrow{\text{hom}} A \right\}$

$$\updownarrow$$

$$
\left\{ \beta, \beta' \text{ as in }
\begin{tikzcd}
H \arrow[r, "f"] \arrow[d, "f"'] & G' \arrow[d, "\beta'"] \\
G \arrow[r, "\beta"'] & A
\end{tikzcd}
\;\middle|\; \beta \circ f = \beta' \circ f' \right\}
$$

In class next week (usual time and place)
exam prep session, NOT recorded